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ABSTRACT

Much research on student understanding of functions has been characterized by a "multi-representational" perspective that investigates students' efforts to make connections among conventionally accepted mathematical representations such as graphs, tables, and equations. In contrast, a "quantitative" perspective explores students' efforts to identify the quantities and relationships between quantities in a function, and a "phenomenological" perspective considers the social and experiential aspects inherent in students' learning experiences. Two research studies have demonstrated ways to incorporate the three research perspectives. One studied how seventh graders develop their understandings of linear functions as they interact with software in the social setting of a traditional classroom. The other study investigated how quantitative reasoning affects the nature of students' generalizations about linear functions and rates of change. Design features of each experiment that explores the three research perspectives are outlined. (Contains 5 figures and 36 references.) (SLD)

Three Perspectives in Research on Functions: Multi-representational, Quantitative, and Phenomenological

Joanne Lobato and Janet Bowers

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Much of the research on students' understandings of functions during the 1980s and 1990s can be characterized by what we call a "multi-representational " perspective. The main focus of this perspective was to investigate students' efforts to make connections among conventionally accepted mathematical representations such as graphs, tables, and equations. This fundamental position has been augmented in recent years to consider two further aspects of understanding: students' efforts to identify the quantities and relationships between quantities in a function, and some of the social and experiential aspects that are inherent in students' learning experiences. We characterize these two perspectives as quantitative and phenomenological, respectively.

The purpose of this paper is to describe and analyze how each these three research perspectives, or some combination of them, can inform our efforts to analyze, and ultimately improve, students' understanding of functions. The paper is organized into two parts. First, the three research perspective are described and analyzed. Second, two research programs that combine elements of the three perspectives with different emphases are compared and contrasted in order to provide the reader with a sense for how these perspectives might guide current research.

Part 1: Description of Three Research Perspectives

Multi-representational Perspective

What is it? We have coined the term "multi-representational perspective" to refer to research (or curriculum design) that identifies conceptual understanding in the domain of mathematical

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functions as the set of connections that students make across representations, typically tables, equations, and graphs — what Nemirovsky, Kaput, & Roschelle (1998) call the “big three". According to Yershalmy & Schwartz (1993), understanding in this view involves "the ability to move nimbly between and to generalize readily between representations is the essence of understanding (p. 45)." Similarly, Moschkovich, Schoenfeld, & Arcavi (1993) conclude their chapter on what it means to understand the domain of linear functions, "We hope to have indicated that competence in the domain consists of being able to move flexibly across representations and perspectives where warranted: to be able to "see" lines in the plane, in their algebraic form, or in tabular form, as objects when any of those perspectives are useful, but also to switch to the process perspective where that perspective is appropriate" (p. 97). From our view, the main tenet of this perspective is that students' understanding can be depicted as forming a web of connections, as shown in Figure 1.

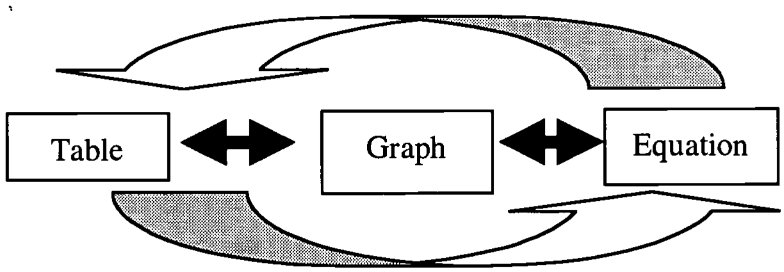

Figure 1. Multi-representational View of Functions

Historical factors contributing to the rise of the multi-representational perspective. Two factors influenced the emergence of the multi-representational perspective. First, technological advances in computer software and hand-held graphing calculators opened up the possibility of linking various representations. Graphs could be produced quickly and easily from a table of values or from an algebraic equation, thus affording opportunities for students to investigate patterns among families of functions without being bogged down by the process of plotting functions by hand. Second, reform documents called for greater attention to conceptual

development across mathematical topics. For example, the Functions standard for high school mathematics in the NCTM *Curriculum and Evaluation Standards* (1989, p. 154) emphasized the multi-representational perspective of understanding functions by stating "… all students should be able to represent and analyze relationships using tables, verbal rules, equations, and graphs; translate among tabular, symbolic, and graphical representations of functions; and analyze the effects of parameter changes on the graphs of functions".

Researchers tried to further flesh out what it means to understand the domain of functions by identifying an important set of concepts involved in linking representations. For example, Schoenfeld, Smith, & Arcavi (1993) identified a fundamental connection that they called the "Cartesian Connection"—a point is on the graph of the line L if and only if its coordinates satisfy an equation of L. They also identified a set of concepts involved in understanding the Cartesian Connection, e.g., for any two points on a line, (x_1, y_1) and (x_2, y_2), $y_2 - y_1$ and $x_2 - x_1$ are directed line segments in the plane, so their ratio determines both the steepness and the orientation of linear graphs. These examples help illustrate what counts as conceptual understanding within the multi-representational perspective.

An oft-cited example from the 4th NAEP demonstrates that researchers and reform document authors had identified concepts that were indeed problematic for students. Specifically, students were asked to draw a line through the origin that is parallel to the line shown and then write an equation of the new line. Researchers assuming a multi-representational approach to developing and analysing this question reasoned that the most likely way to solve this task is to use the following logical progression of concepts involving connections between graphs and equations:

- The m value in linear equations of the form $y = mx + b$ represents the slope;
- the slopes of two parallel lines are the same, since the slope measures the steepness and direction of the line; the b value is the y-intercept on the graph; and
- the b value for the new equation will be 0 since the new line goes through the origin.

Only 16% of twelfth graders answered both parts of this question correctly. Assuming a multi-representational perspective, one could conclude that 84% of the population tested did not really *understand* the concept of function because they were unable to "see" the connections between the graphical and algebraic representations.

Critique. The major theme of this symposium is that the multi-representational perspective in research is indispensable, and we cannot afford to exclude it. However, we believe that two major concerns, which have become apparent with the emergence of quantitative and phenomenological perspectives, need to be addressed.

The first concern is whether tables, graphs, and equations are multiple representations of *anything* to students. That is, students may learn to move among the representations, but not understand what is being represented. In other words, from the students' point of view, the core concept of "function" may not represented by any of what are commonly called the multiple representations of functions. According to Thompson (1994a), an alternative focus is to consider whether students who are making connections among multi-representational activities experience a subjective sense of invariance. Within a quantitative perspective, the object of representation can be conceived as a relationship among quantities in a situation, and tables, graphs, or equations are simply different ways to represent that relationship. This quantitative perspective will be further elaborated shortly.

The second concern is that the multi-representational perspective tends to overlook the roles of students' everyday experience, kinesthetic experience, natural language, and the social negotiation in the development of meaning. Research from the phenomenological perspective broadens the psychological approach to include the social realm and the relationship between students' experiential worlds and school mathematics or science (diSessa, 1993).

Quantitative Perspective

What is it? Research from the quantitative perspective involves an exploration of the processes of analyzing quantities and relationships among quantities in situations and creating new quantities. Quantities are constituted in people's conceptions about measurable attributes of objects, events, or situations.[1]For example, we might think about heights of people we know, the distance we travel driving to work, or how old we'll be in 15 years. Quantities are not numbers, though numbers can be values of quantities, e.g., 6 feet or 40 years.[2]

Many issues become the focus of research in a quantitative perspective that have not be important in a multiple-representational approach. For example, Simon and Blume (1994) and Lobato and Thanheiser (1999) investigate the process of creating ratios and rates as measures of particular qualities like motion through space of a person running or the steepness of a ramp. Harel, Behr, Lesh & Post (1994) examined the concept of invariance of ratio through an investigation of students' understanding of constancy of taste (i.e., the notion that random samples of a mixture like orange juice or coca cola will taste the same) using methods that did not rely on conventional symbolism or on reasoning with numerical values.

A quantitative perspective goes beyond simply studying the use of "real world" problem situations or using graphs and equations to model phenomena. For example in Thompson's (1994b) examination of students' conceptions of rate, he investigated students' construction of distance and time as extensive quantities and speed as an intensive quantity formed by a multiplicative relationship between distance and time (as opposed to studying how students were interpreting graphs in terms of speed situations).

[1] Thompson (1994) defines a quantity as a conceptual entity that involves an object, an attribute or quality of the object, an appropriate unit, and a process of assigning a numerical value to the quality.
[2] Although Kaput extends Thompson's approach by distinguishing between situation-based quantitative reasoning and abstract quantitative reasoning, we will focus on the former category for this paper.

How do quantitative and multi-representational perspectives differ? Quantitative and multi-representation perspectives are compatible in principle. In fact, quantitative reasoning can be conceived as foundational to an understanding of the relationships that are being represented by conventional forms like equations or graphs. In the words of Thompson & Thompson (1995):

> *From our perspective, to ground the development of algebraic thinking on the notion of functions and functional relationships without, in turn, grounding these on understandings of quantities and quantitative reasoning in dynamic situations, is like building a house starting with the second floor. The house will not stand. (p. 98).*

However when the object perspective of functions in the multi-representational approach (see Kieran, 1993 or Moschkovich, Arcavi, and Schoenfeld, 1993) is conceived (either by researchers or by teachers and students) as a *physical object* (e.g., thinking of a line on a graph as a piece of wire) rather than as a *reified mathematical object* (Sfard, 1994), then some incompatibilities between quantitative and multi-representational perspectives arise. For example, Magidson (1992) describes an activity in which students are asked to recreate the picture of a Starburst (shown in Figure 2) by entering different values for *m* into the equation y = mx, using graphing software or a graphing calculator. Students are encouraged to experiment and note patterns between values of *m* and behavior of the lines. The activity is designed to address the connections between graphs and equations as described in the left hand column of Figure 3. In contrast, it is interesting to assume a quantitative perspective in order to postulate a related set of math goals aimed at understanding functions as relationships among quantities. The differences between these perspectives is highlighted in the right hand column in Figure 4.

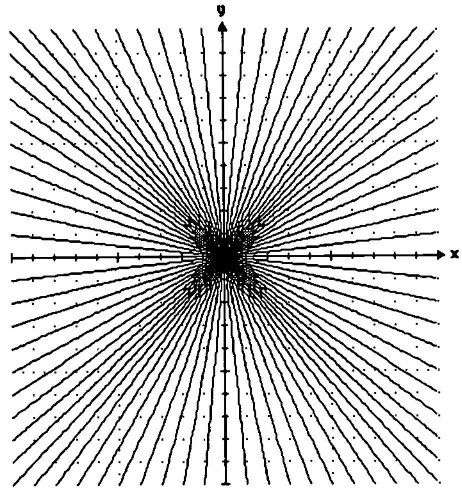

Figure 2. Re-create the Starburst

Math Concepts In a Multi-representational Perspective	**Math Concepts in a Quantitative Perspective**
Lines with a positive value for *m* rise to the right Lines with negative value for *m* fall to the right	Lines with a positive value for *m* represent functions where both quantities are increasing or both quantities are decreasing Lines with a negative value for m represent functions where one quantity is increasing while the other is decreasing
The *m* values control the steepness and direction of the line	Slope is a measure of some attribute in a situation (like velocity, density, sweetness, etc) that is affected by various quantities (e.g., distance and time in the case of velocity)
Various *m* values place the lines in different quadrants on the graph	Slope is a rate of change of one quantity in terms of another quantity, where both quantities co-vary

Figure 3. Contrasting mathematical concepts emphasized
in multi-representational versus quantitative perspectives

While Figure 3 primarily demonstrates differences in focus between quantitative and multi-representational perspectives, the differences in the second row in the chart can lead to an incompatibility. Specifically, if slope is conceived of as a characteristic of a line as object, i.e., as

a measure of that line's steepness, then two lines with the same slope should have the same steepness. However, Figure 4 illustrates a counterexample. Of course, one can patch this difficulty by saying that slope is a measure of the steepness within the same scale system, but that doesn't really resolve the underlying problem. Alternatively, slope can be conceived as the rate of change in one quantity relative to the change of another quantity, where the two quantities co-vary. Then slope is a measure of an attribute (like velocity, dripping rate of a leaky faucet, or gas efficiency) that is affected by the quantities in a functional relationship. In the examples shown in Figure 5, the slope of each line represents the rate at which the water is flowing into a measuring cup rather than a measure of the steepness of the line.

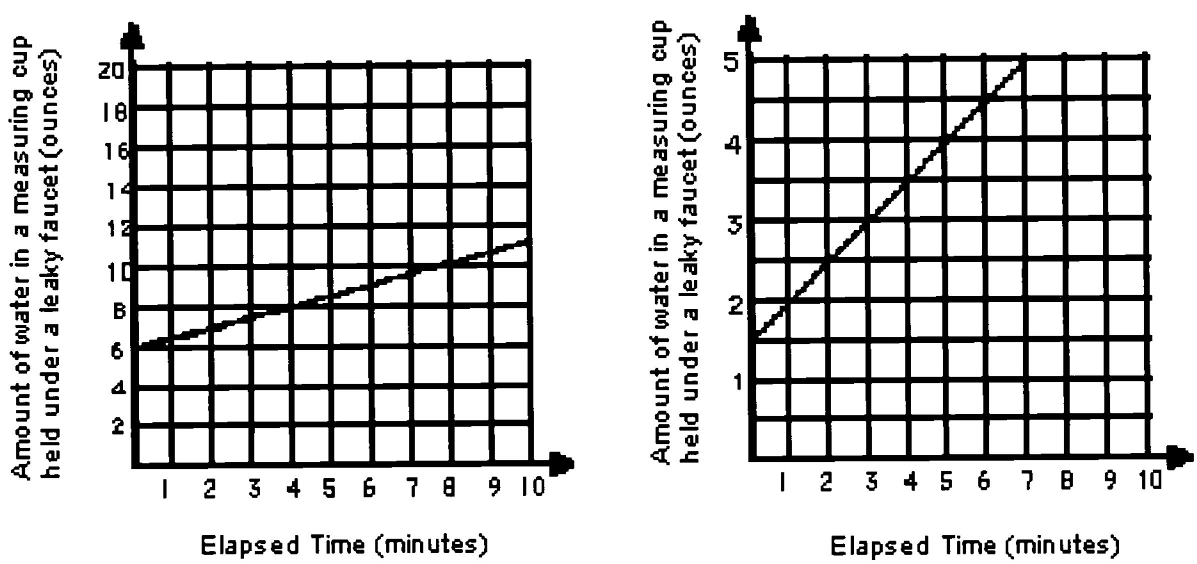

Figure 4: Two lines with the same slope but not the same steepness

Phenomenological Perspective

What is it? The roots of the phenomenological perspective in mathematics education may be traced to the work of phenomenological anthropologists. According to Laughlin (1994), phenomenological anthropologists have been "...concerned with methods that may be utilized in fieldwork to 'get into the native's head' and understand what the native is experiencing"(p. 924).

One of the most profound tenets of this field of work is that, in contrast to structuralists who claim that texts (or works, or diagrams, or other forms of expression) can be interpreted independently from the actual experiences of the people who create them, phenomenological anthropologists maintain that accurate interpretations of native works require "accessing the experiences of living people who are influenced by the text" (Laughlin, p. 925).

For mathematics education researchers assuming a phenomenological perspective, the implication of the above-described tenet is that any analysis involves attempting to explain the dialogic relationship between the text or artifact and the experiences of the people who are participating in its use. Analyzing this dialogic experience involves recognizing a reflexive process in which meanings are developed within the social context of the participants and then re-interpreted as the participants reflect on their experiences and reorganize their conceptions within the community dialog (cf. Bowers & Nickerson, 1998; Cobb, et al.,1997; Sfard, 2000). In this way, the goal of the analysis is to study students' interpretations of their experiences of the activities within their own social contexts in such a way as to reconstruct the "real" meaning of the activity in terms of how they exist and are interpreted in the native culture.

Recent studies on functions from the phenomenological perspective have addressed the following issues: how students' kinesthetic experiences are related to their emerging understanding of functions and graphs (Nemirovsky, Tierney & Wright, 1998; Nemirovsky & Monk, 2000); how scientific practices influence physicists' interpretations of graphs (Ochs, et. al, 1994); the social negotiation of meaning of key terms like "slope" and "steepness" (Moschkovich, 1996); and the role that natural language plays in the development of intensive quantities (Lobato & Thanheiser,1999). All of these works include some phenomenological tenets in their design. First, they are all focused on "getting into the native's head" in that they attempt to frame their subjects' interpretations in terms of their own experiences, rather than from an [ostensibly] objective perspective. Second, they all focus on describing the *process* of

coming to know, rather than describing only the *product* that results from personal reflection and reorganization.

How do the three perspectives differ? One way to characterize the differences among these perspectives is to focus on their historical evolutions and the questions they were developed to answer. As noted, the multi-representational view arose during a period when technological innovations were being introduced into schools. The climate of the times demanded a framework that could address the effectiveness of the tools, and researchers rushed to develop a variety of dynamic and linked representation systems to document students' increasingly integrated views of mathematics as a result of their experiences with integrated or linked representations. In brief, the unit of analysis was a student's ability to produce one representation given another, and his ability to choose among the conventional representations judiciously.

The quantitative perspective can be seen as arising in reaction to this focus on relations between mathematical *representations*. In essence, this perspective calls for a focus on students' understanding of the quantities (and their relations) that the representations portend to signify. Therefore, the unit of analysis has been each individual student's description of how he or she is thinking about the relationships between the quantities, and how he or she measures the attribute in question, rather than focusing on the student's interpretation of how a conventional representation might quantify this relation.

Although these two perspectives differ, they both share a focus on how well an individual student's conceptions align with the "actual" mathematical relations represented in the inscription or situation. On the one hand, the multiple representations approach assumes an *observer's* perspective to describe the proximity between the student's current views of the representations and their culturally-accepted use. On the other hand, Thompson (1994b) describes several conceptual interpretations students develop as they begin to conceptualize

speed as a rate. In this sense, he attempts to describe the actor's perspective to the extent of determining the mathematical imagery on which she is creating her interpretations.

The phenomenological perspective focuses exclusively on describing the actor's *experience* by capturing the students' interpretations as they occur, and wherever they lead. This differs from framing them as steps along a path with a known goal (such as toward a conception of speed as a rate). For example, Nemirovsky and Monk (2000) describe the constructs of fusion and trail blazing to convey their subject's ongoing re-interpretations of the graphs formed by a motion detector. In their analysis, the authors analyze the processes through which two students come to interpret certain linear representations by framing their experiences in their "lived-in space" rather than in the metaphorical space of formal mathematics. It is hopefully clear that, in our view, the phenomenological approach builds on the former two perspectives, but attempts to portray a broader picture of the learning process rather than focusing in an in-depth way on either a) students' connections among algebraic symbolizations, or b) students' views of the relationships between the quantities being represented.

As Hiebert (1999) notes, research perspectives should be chosen based on their own values and questions they wish to pursue. We, too, believe that this is a crucial distinction to make and are not putting forth any one perspective as inherently better than another. In fact, our differing questions have led us to emphasize different aspects of these three perspectives in our work. In what follows, we delineate these choices in terms of our research goals and programs.

Part 2. Two Different Ways to Incorporate the Three Research Perspectives

A. Bower's Approach

The research question that my research team and I have explored involved conducting a teaching experiment in a 7th grade classroom with students using SimCalc (a microworld with linked representations designed to help students experience some of the fundamental ideas of

calculus). The question we were investigating was, "How do students in a traditional classroom (with its external pressures for curricular coverage and prescribed topics) come to develop understandings of linear functions as they interact with the software in the social setting of the classroom?" (cf., Bowers & Nickerson, 1999; Nickerson, Nydam, & Bowers, in press). One of the most striking aspects of this question is its length. It would seem so much more straight-forward to simply ask, "What was the effect of the computer software?" The reason that we did not frame the question in this way is that we do not believe that the unit of analysis can be a software program itself. As noted above, from a phenomenological perspective, an accurate interpretation of an artifact cannot be determined without considering the entire range of experiences of the living people who are interpreting it. In other words, it is not the influence of any one object, but instead the situated experience we try to capture. Moreover, given our view that learning is an inherently social process, we wanted to know if it was possible to make a replacement unit that would cover the same material as a textbook, but engage students in novel experiences in which they could develop interpretations for conventional algebraic representations (the big three) in ways that paper and pencil could not afford.

We follow Kilpatrick (1981) in believing that "In a non-trivial sense, the results are the least important aspect of a research study… The most important aspect of a research study is the constructs and theories used to interpret the data" (Kilpatrick, cited in Sowder, 2000). In what follows, I describe a research construct, rather than a specific result, that was derived from aspects of each of the three perspectives under discussion.

As noted earlier, the broad question we were addressing in the study was how 7^{th} grade students who were using SimCalc-based activities developed understandings of linear functions. Although this software was initially designed to engage students in the broad ideas of calculus, it also lends itself to exploring how algebraic relations can be used to record motion. The microworld consists of an animation world in which cartoon characters (such as frogs, aliens, and ducks) move about in one directional motion paths, and three graphical worlds (position *v.*

time, velocity *v.* time, and acceleration *v.* time), which can all be linked to the animated simulation and to each other. In order to exploit these links, we designed instructional activities that would encourage students to "think before they click", or hypothesize what might happen to one representation when another linked representation is changed. Following Arcavi (1999), we anticipated that the prediction aspect of the activity might encourage the students to be clearer about how they envision the situation, engage them in the practice of anticipating and making predictions among representations by reasoning through the phenomenon, and create more interest and excitement in watching the resulting simulation. If they were not correct, we hoped that a perturbation would arise and that they would revise their sketches by asking, "What if … questions" to see if they could explain the differences between what they expected and what they saw.

What we found was that students did *not* tend to make predictions, and hence did not investigate or try to resolve any perturbations. In order to challenge their ways of working, we developed other activities that focused on developing this "what if propensity". In so doing, we noted that, in this traditional setting, the linked representations did not serve the goal of helping students to see the "links" between position and velocity. In fact, they originally *discouraged* students from asking such questions because the students could see that the computer had already linked these graphs, and hence there was nothing to formulate in order to achieve their goals. In short, they viewed their role as computer users who had to guess how outcomes were determined in the "black box" rather than trying to figure out the mathematics hidden inside the box. Through our focus on the development of a "what if propensity" in whole-class discussions and in activity design, the students did come to reorganize their methods from clicking and then thinking, to thinking *before* clicking. Moreover, our data indicates that they also came to reorganize their views of what it meant to engage in mathematical explorations (cf., Bowers & Nickerson, 1999; and Bowers & Kennehan, submitted for a further description of these data).

How was this work informed by each of the perspectives?

The SimCalc software design itself, which was created by Kaput and Roschelle in the early 1990s, is derived from Kaput's variation of the multi-representational view. Whereas most function graphing software that includes multiple, linked representations of *the same data set,* Kaput's central focus of the MathWorlds software is the exploration of *the same phenomena* based on viewing representations of *different data sets* (i.e., position, velocity, and acceleration). This design, which is intended to place the animation (i.e., the phenomenon) at the center of the students' activity, is based on Kaput and Roschelle's (1997) attempt to address the "island problem" wherein students who may be able to move among mathematical representations (e.g., graphs, tables, and algebraic expressions) within the mathematical island of formalisms may still not be making connections with the mainland of human experience (Kaput, 1994).

The design and analysis of this study also drew on the phenomenological perspective in that the constructs used in the analysis were developed to coordinate the students' social and cognitive experiences with the software. The construct of the "What if propensity" was developed to document the ways in which students came to value the use of anticipatory thinking and reflection in order to revise their hypotheses more efficiently than simply "clicking without thinking."

Although used to a lesser extent, the study did draw on the quantitative perspective's emphasis on supporting students' views of the relations between the quantities. While we viewed this as a critical aspect of coming to know and understand students' interpretations of graphs, we chose to frame our activities to "cover" the same topics as would have been covered if we had not developed the replacement unit. Thus, we were forced to focus on students' understandings of linear graphs and algebraic equations rather than on their conceptions of rate that might have developed separately from the conventional algebraic inscriptions with which they were working. In other words, although we conducted pre and post interviews in which we asked students to describe quantities such as rate, we could not separate their understanding of

the relations between the quantities from their understandings of the representations because they were experienced simultaneously. For example, when students were asked questions such as, "How would the speed of a runner change if she ran two-thirds of the distance in twice the amount of time?", they would often reason "through the graph" by noting that they would "decrease the slope of the line [on the position graph]". In our analysis, we attempted to discern whether these types of explanations reflected the students' actual thinking processes, or whether they reflected the students' views of what we, as familiar classroom observers and interviewers, expected and counted as acceptable explanations. In the following section, Joanne considers the possibility of waiting to introduce graphical notations until after students have developed an image of the quantities involved.

B. Lobato's Approach

The goal of my research project, *The Generalization of Learning In Multimedia Environments*, is to offer a theoretical reconceptualization of transfer. In service of this goal, members of my research team and I have been investigating how quantitative reasoning can affect the nature of students' generalizations about linear functions and rates of change.

Our motivation for focusing on quantitative reasoning arises, in part, from previous studies where we found that classic transfer tasks for slope (e.g., asking students to find the slope of a playground slide or wheelchair ramp) did not represent simple applications for students but rather were problematic because students did not understand the quantitative complexity of each situation (Lobato, forthcoming). In two follow-up studies, we found that over half of the Algebra 1 students in the sample (n = 17) had difficulty understanding the effect that increasing or decreasing the base or the platform of a wheelchair ramp had on the steepness of that ramp (Lobato & Thanheiser, 1999; Lobato, 1999; Lobato, in preparation). Students also had difficulty focusing on the attribute to be measured in a situation. For example, when asked to create a way

to measure how fast a mouse was traveling, some students had trouble isolating the attribute of how fast one's legs are moving from how fast an object is moving through space. Similarly, over half the students had difficulty isolating the attribute of steepness in the wheelchair ramp situation from attributes like 'work required to climb the ramp" or "materials needed to construct a ramp."

We hypothesized that if students understood the relationships between quantities and were able to form multiplicative relationships between covarying quantities that the nature of their generalizations about rates of change would be more productive. Thus, we conducted one teaching experiment last summer and are currently conducting a second teaching experiment in order to help high school students reason quantitatively and to study their generalizations. We are making progress in helping students reason quantitatively, though we have found this goal to be extremely challenging. As Thompson & Thompson (1995) wrote, "Students' quantitative reasoning is almost an oxymoron since students do not, for the most part, reason quantitatively in mathematics classrooms. Instead their thinking is dominated by numeric calculations, which are often not connected to the quantities and relationships among quantities that might be represented by the calculations."

The following three design features of our teaching experiments illustrate our approach toward quantitative and multi-representational perspectives on functions:

1. We spend substantial periods of time developing quantitative reasoning without any appeal to conventional representations like graphs or equations. In fact, during the 30 hours of instruction in our first teaching experiment and in the 12 hours that we have conducted during our second and ongoing teaching experiment, we have not yet introduced graphs or equations. For example, in our use of SimCalc's Mathworlds computer software (J. Kaput, PI), we "turn off" the graphical features and use it as an simulation environment for exploring relationships between the quantities that affect velocity.

2. We had to re-think the "big ideas" of understanding slope and linear functions with a focus on quantitative reasoning (Lobato, in preparation). For example, the following three mathematical goals guide the beginning work of the teaching experiment:
   - isolate measurable attributes (e.g., how fast one's legs are moving from how fast the character is moving through space)
   - understand which quantities affect the attribute of interest and in what ways (e.g., what quantities are essential to measure if one wants to measure how fast a character is walking; how does increasing or decreasing the distance one walks but fixing time affect one's velocity)
   - construct a ratio-as-measure (as a composition of composition numbers or as a multiplicative comparison) and build an equivalence class of ratios (perhaps through iterating and partitioning the ratio)
3. We plan on eventually introducing the conventional representations of graphs and equations once students have constructed rate relationships between quantities in situations. This does not mean that representations do not play a critical role in our current work. Currently we are studying the evolution of non-standard student-constructed representations, like diagrams or tables that lack the sequential ordering of the tables that are often presented to students in school. Thus, representations play an important role in our approach even though we are not currently working from a multi-representational perspective.

In view of the three perspectives, our instructional approach begins with a quantitative perspective with plans to eventually help students represent quantitative relationships with conventional representations and move flexibly among those representations. Our analysis focuses on students emerging quantitative reasoning and the types of generalizations that arise as a result. The phenomenological perspective also enters into our analysis. We are particularly

interested in understanding the experiential basis for students' comprehension of relationships among quantities. For example, in the current teaching experiment, all four pairs (n = 8) have, at one time, focused on time exclusively as a measure of how fast one travels. This notion seems to have a basis in students' common everyday experience of determining the speed of runners. In a race, the distance is fixed; thus one can simply measure how fast a runner travels by timing the race. Distance seems to be an implicit quantity for many students, in this situation. Another common everyday experience seems to be that of being a child trying to keep up with a parent. The child feels a sensation of going faster than the parent, in order to keep up. Consequently, we see students in our current teaching experiment mention this experience when they suggest that quantities like "number of steps" affect how fast one travels. There is much to study in terms of the social negotiation of terms like "speed" and "how fast one goes." One pair of students in the teaching experiment had a heated argument over several sessions about how speed was not a measure of how fast one travels. They seemed to want different terms for each of the following quantities: 1) a measure of how fast an object travels through space; 2) a measure of how fast one's legs move when walking; and 3) a measure of how fast runners travel in a race (since only time seems to matter there). Figure 5 summarizes how we have combined the three research perspectives in our work.

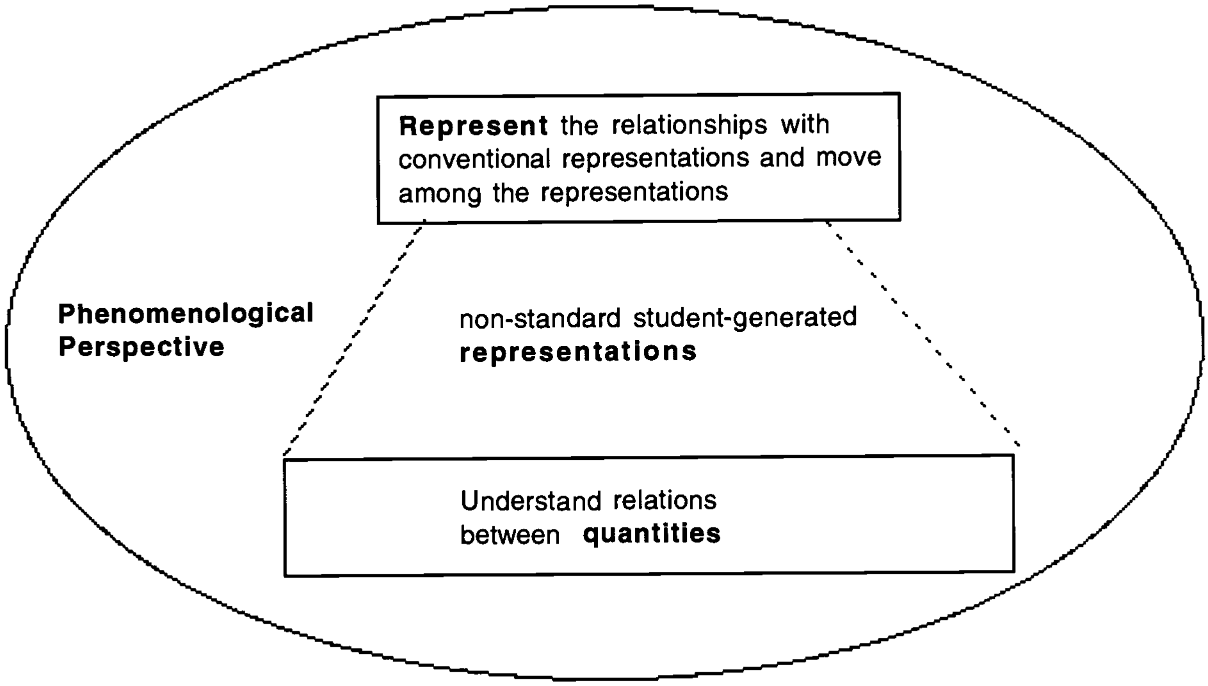

Figure 5. Combination of 3 perspectives in Lobato's work

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